

# NEW SHARP CUSA–HUYGENS TYPE INEQUALITIES FOR TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

ABSTRACT. We prove that for  $p \in (0, 1]$ , the double inequality

$$\frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2} < \frac{\sin x}{x} < \frac{1}{3q^2} \cos qx + 1 - \frac{1}{3q^2}$$

holds for  $x \in (0, \pi/2)$  if and only if  $0 < p \leq p_0 \approx 0.77086$  and  $\sqrt{15}/5 = p_1 \leq q \leq 1$ . While its hyperbolic version holds for  $x > 0$  if and only if  $0 < p \leq p_1 = \sqrt{15}/5$  and  $q \geq 1$ . As applications, some more accurate estimates for certain mathematical constants are derived, and some new and sharp inequalities for Schwab-Borchardt mean and logarithmic means are established.

## 1. INTRODUCTION

The Cusa and Huygens (see, e.g., [1]) states that for  $x \in (0, \pi/2)$ , the inequality

$$(1.1) \quad \frac{\sin x}{x} < \frac{2 + \cos x}{3}$$

holds true. Its version of hyperbolic functions refers to (see [2]) the inequality

$$(1.2) \quad \frac{\sinh x}{x} < \frac{2 + \cosh x}{3}$$

holds for  $x > 0$ , and it is known as hyperbolic Cusa–Huygens inequality (see [2]).

There are many improvements, refinements and generalizations of (1.1) and (1.2), see [3], [4], [5], [6], [7], [8], [9]; [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21].

Now we focus on the bounds for  $(\sin x)/x$  in terms of  $\cos px$ , where  $x \in (0, \pi/2)$ ,  $p \in (0, 1]$ . In 1945, Iyengar [22] (also see [23, subsection 3.4.6]) proved that for  $x \in (0, \pi/2)$ ,

$$(1.3) \quad \cos px \leq \frac{\sin x}{x} \leq \cos qx$$

holds with the best possible constants

$$p = \frac{1}{\sqrt{3}} \quad \text{and} \quad q = \frac{2}{\pi} \arccos \frac{2}{\pi}.$$

Moreover, the following chain of inequalities hold:

$$(1.4) \quad \cos x \leq \frac{\cos x}{1 - x^2/3} \leq (\cos x)^{1/3} \leq \cos \frac{x}{\sqrt{3}} \leq \frac{\sin x}{x} \leq \cos qx \leq \cos \frac{x}{2} \leq 1.$$

Qi et al. [24] showed that

$$(1.5) \quad \cos^2 \frac{x}{2} < \frac{\sin x}{x}$$

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holds for  $x \in (0, \pi/2)$ . Klén et al. [25, Theorem 2.4] pointed out that the function  $p \mapsto (\cos px)^{1/p}$  is decreasing on  $(0, 1)$  and for  $x \in \left(-\sqrt{27/5}, \sqrt{27/5}\right)$

$$(1.6) \quad \cos^2 \frac{x}{2} \leq \frac{\sin x}{x} \leq \cos^3 \frac{x}{3} \leq \frac{2 + \cos x}{3}$$

are valid. Subsequently, Yang [8] (also see [26]) gave a refinement of (1.6), which states that for  $p, q \in (0, 1)$  the double inequality

$$(1.7) \quad (\cos px)^{1/p} < \frac{\sin x}{x} < (\cos qx)^{1/q}$$

holds for  $x \in (0, \pi/2)$  if and only if  $p \in [p_0^*, 1)$  and  $q \in (0, 1/3]$ , where  $p_0^* \approx 0.3473$ . Moreover, the double inequality

$$\left(\cos \frac{x}{3}\right)^\alpha < \frac{\sin x}{x} < \left(\cos \frac{x}{3}\right)^3$$

with the best exponents  $\alpha = 2(\ln \pi - \ln 2) / (\ln 4 - \ln 3) \approx 3.1395$  and 3. Also, he pointed out that the value range of variable  $x$  such that (1.6) holds can be extended to  $(0, \pi)$ . Very recently, Yang [21] gave another improvement of (1.6), that is, for  $x \in (0, \pi/2)$  the inequalities

$$(1.8) \quad \frac{\sin x}{x} < \left(\frac{2}{3} \cos \frac{x}{2} + \frac{1}{3}\right)^2 < \cos^3 \frac{x}{3} < \frac{2 + \cos x}{3}$$

are true.

An important improvement for the inequality in (1.5) is due to Neuman [2]:

$$(1.9) \quad \cos^{4/3} \frac{x}{2} = \left(\frac{1 + \cos x}{2}\right)^{2/3} < \frac{\sin x}{x}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Lv et al. [27] showed that for  $x \in (0, \pi/2)$  inequalities

$$(1.10) \quad \left(\cos \frac{x}{2}\right)^{4/3} < \frac{\sin x}{x} < \left(\cos \frac{x}{2}\right)^\theta$$

hold, where  $\theta = 2(\ln \pi - \ln 2) / \ln 2 = 1.3030\dots$  and  $4/3$  are the best possible constants. By constructing a decreasing function  $p \mapsto (\cos px)^{1/(3p^2)}$  ( $p \in (0, 1]$ ), Yang [9] showed that the double inequality

$$(1.11) \quad (\cos p_1^* x)^{1/(3p_1^{*2})} < \frac{\sin x}{x} < \cos^{5/3} \frac{x}{\sqrt{5}}$$

is true for  $x \in (0, \pi/2)$  with the best constants  $p_1^* \approx 0.45346$  and  $1/\sqrt{5} \approx 0.44721$ . It follows that

$$(1.12) \quad \begin{aligned} (\cos x)^{1/3} &< \cos^{1/2} \frac{\sqrt{6}x}{3} < \cos^{2/3} \frac{x}{\sqrt{2}} < \cos \frac{x}{\sqrt{3}} < \cos^{4/3} \frac{x}{2} < (\cos p_1^* x)^{1/(3p_1^{*2})} \\ \frac{\sin x}{x} &< \cos^{5/3} \frac{x}{\sqrt{5}} < \cos^2 \frac{x}{\sqrt{6}} < \cos^3 \frac{x}{3} < \cos^{16/3} \frac{x}{4} < e^{-x^2/6} < \frac{2+\cos x}{3} \end{aligned}$$

are valid for  $x \in (0, \pi/2)$ .

For the bounds for  $(\sinh x)/x$  in terms of  $\cosh px$ , it is known that the inequalities

$$\frac{\sinh x}{x} < \cosh^3 \frac{x}{3} < \frac{2 + \cosh x}{3}$$

holds true for  $x > 0$  (see [18]), which is exactly derived by the inequalities for means

$$L < A_{1/3} < \frac{2G + A}{3}$$

(see e.g. [28], [29], [30]), where  $L$ ,  $A_p$ ,  $G$  and  $A$  stand for the logarithmic mean, power mean of order  $p$ , geometric mean and arithmetic mean of positive numbers  $a$  and  $b$  defined by

$$\begin{aligned} L &\equiv L(a, b) = \frac{a - b}{\ln a - \ln b} \text{ if } a \neq b \text{ and } L(a, a) = a, \\ A_p &\equiv A_p(a, b) = \left( \frac{a^p + b^p}{2} \right)^{1/p} \text{ if } p \neq 0 \text{ and } A = A_0(a, b) = \sqrt{ab}, \end{aligned}$$

$G = A_0$  and  $A = A_1$ , respectively. Zhu in [31] proved that for  $p > 1$  or  $p \leq 8/15$ , and  $x \in (0, \infty)$ , the inequality

$$\left( \frac{\sinh x}{x} \right)^q > p + (1 - p) \cosh x$$

is true if and only if  $q \geq 3(1 - p)$ . It follows by letting  $p = 1/2$  and  $q = 3/2$  that

$$\frac{\sinh x}{x} > \cosh^{4/3} \frac{x}{2}$$

holds for  $x > 0$  (also see [2, (2.8)]). Yang [19] showed that the inequality

$$(1.13) \quad \frac{\sinh x}{x} > (\cosh px)^{1/(3p^2)}$$

holds for all  $x > 0$  if and only if  $p \geq 1/\sqrt{5}$  and its reverse holds if and only if  $0 < p \leq 1/3$ . And, the function  $p \mapsto (\cosh px)^{1/(3p^2)}$  is decreasing on  $(0, \infty)$ .

The aim of this paper is to determine the best  $p$  such that the inequalities

$$\begin{aligned} \frac{\sin x}{x} &< (>) \frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2}, \quad p \in (0, 1), \quad x \in (0, \pi/2), \\ \frac{\sinh x}{x} &< (>) \frac{1}{3p^2} \cosh px + 1 - \frac{1}{3p^2}, \quad p, x \in (0, \infty) \end{aligned}$$

hold true.

Our main results are contained in the following theorems.

**Theorem 1.** For  $p \in (0, 1]$  and  $x \in (0, \pi/2)$ , the double inequality

$$(1.14) \quad \frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2} < \frac{\sin x}{x} < \frac{1}{3q^2} \cos qx + 1 - \frac{1}{3q^2}$$

holds if and only if  $0 < p \leq p_0 \approx 0.77086$  and  $0.77460 \approx \sqrt{15}/5 = p_1 \leq q \leq 1$ , where  $p_0$  is the unique root of the equation

$$(1.15) \quad F_p\left(\frac{\pi}{2}^-\right) = \frac{2}{\pi} - \left( \frac{1}{3p^2} \cos \frac{p\pi}{2} + 1 - \frac{1}{3p^2} \right) = 0$$

on  $(0, 1)$ . And, the bound for  $(\sin x)/x$  given in (1.14) is increasing with respect to parameters  $p$  or  $q$ .

**Theorem 2.** For  $p, x > 0$ , the double inequality

$$(1.16) \quad \frac{1}{3p^2} \cosh px + 1 - \frac{1}{3p^2} < \frac{\sinh x}{x} < \frac{1}{3q^2} \cosh qx + 1 - \frac{1}{3q^2}$$

holds if and only if  $0 < p \leq p_1 = \sqrt{15}/5$  and  $q \geq 1$ . And, the bound for  $(\sinh x)/x$  given in (1.16) is increasing with respect to parameters  $p$  or  $q$ .

**Remark 1.** The weighted basic inequality of two positive numbers of  $a$  and  $b$  tell us that for  $\alpha \in [0, 1]$ , the inequality  $\alpha a + (1 - \alpha)b \geq a^\alpha b^{1-\alpha}$ . It is reversed if and only if  $\alpha \geq 1$  or  $\alpha \leq 0$  (see [32]). Hence, taking into account (1.11) and (1.14) we see that

(i) if  $p \in [p_1^*, 1/\sqrt{3})$ , where  $p_1^* \approx 0.45346$ , then

$$\frac{\sin x}{x} > (\cos px)^{1/(3p^2)} > \frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2};$$

(ii) if  $p \in (1/\sqrt{3}, p_0)$ , where  $p_0 \approx 0.77086$ , then

$$\frac{\sin x}{x} > \frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2} > (\cos px)^{1/(3p^2)}.$$

In the same way, (1.13) together with (1.16) leads us to

(iii) if  $p \in [1/\sqrt{5}, 1/\sqrt{3})$ , then

$$\frac{\sinh x}{x} > (\cosh px)^{1/(3p^2)} > \frac{1}{3p^2} \cosh px + 1 - \frac{1}{3p^2};$$

(iv) if  $p \in (1/\sqrt{3}, \sqrt{15}/5)$ , then

$$\frac{\sinh x}{x} > \frac{1}{3p^2} \cosh px + 1 - \frac{1}{3p^2} > (\cosh px)^{1/(3p^2)}.$$

Taking  $p = 3/4, 1/\sqrt{2}, 2/3, 1/\sqrt{3}$  and  $q = \sqrt{3/5}, \sqrt{2/3}, \sqrt{3}/2, 1$  in Theorem 1, we have

**Corollary 1.** For  $x \in (0, \pi/2)$ , the inequalities

$$\begin{aligned} (1.17) \quad \cos \frac{x}{\sqrt{3}} &< \frac{3}{4} \cos \frac{2x}{3} + \frac{1}{4} < \frac{2}{3} \cos \frac{x}{\sqrt{2}} + \frac{1}{3} < \frac{16}{27} \cos \frac{3x}{4} + \frac{11}{27} < \frac{\sin x}{x} \\ &< \frac{5}{9} \cos \frac{\sqrt{15}x}{5} + \frac{4}{9} < \cos^2 \frac{x}{\sqrt{6}} < \frac{4}{9} \cos \frac{\sqrt{3}x}{2} + \frac{5}{9} < \frac{1}{3} \cos x + \frac{2}{3}. \end{aligned}$$

Putting  $p = \sqrt{3/5}, 3/4, 1/\sqrt{2}, 2/3, 1/\sqrt{3}$  and  $q = 1, 2/\sqrt{3}$  in Theorem 2 we have

**Corollary 2.** For  $x > 0$ , the inequalities

$$\begin{aligned} (1.18) \quad \cosh \frac{x}{\sqrt{3}} &< \frac{3}{4} \cosh \frac{2x}{3} + \frac{1}{4} < \frac{2}{3} \cosh \frac{x}{\sqrt{2}} + \frac{1}{3} < \frac{16}{27} \cosh \frac{3x}{4} + \frac{11}{27} \\ &< \frac{5}{9} \cosh \frac{\sqrt{15}x}{5} + \frac{4}{9} < \frac{\sinh x}{x} < \frac{1}{3} \cosh x + \frac{2}{3} < \frac{1}{2} \cosh^2 \frac{x}{\sqrt{3}} + \frac{1}{2}. \end{aligned}$$

## 2. PROOF OF THEOREM 1

In order to prove Theorem 1, we need some lemmas.

**Lemma 1.** For  $x \in (0, \pi/2)$ , the function  $p \mapsto U_p(x)$  defined on  $[0, 1]$  by

$$U_p(x) = \frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2} \text{ if } p \in (0, 1] \text{ and } U_0(x) = 1 - \frac{x^2}{6}$$

is increasing.

*Proof.* Differentiation yields

$$\begin{aligned} \frac{\partial U_p}{\partial p} &= \frac{1}{3p^3} (2 - 2 \cos px - px \sin px) \\ &= \frac{12px}{3p^3} \left( \frac{\sin \frac{px}{2}}{\frac{px}{2}} - \cos \frac{px}{2} \right) \sin \frac{px}{2} > 0, \end{aligned}$$

which completes the proof.  $\square$

**Lemma 2.** *Let the function  $F_p$  be defined on  $(0, \pi/2)$  by*

$$(2.1) \quad F_p(x) = \frac{\sin x}{x} - \left( \frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2} \right) \text{ if } p \in (0, 1] \text{ and } F_0(x) = \frac{\sin x}{x} - 1 + \frac{x^2}{6}.$$

(i) *If  $F_p(x) < 0$  for all  $x \in (0, \pi/2)$ , then  $p \in [p_1, 1]$ , where  $p_1 = \sqrt{15}/5 \approx 0.77460$ .*

(ii) *If  $F_p(x) > 0$  for all  $x \in (0, \pi/2)$ , then  $p \in [0, p_0]$ , where  $p_0 \approx 0.77086$ .*

*Proof.* (i) If  $F_p(x) < 0$  for all  $x \in (0, \pi/2)$ , then we have

$$(2.2) \quad \lim_{x \rightarrow 0} \frac{\frac{\sin x}{x} - \left( \frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2} \right)}{x^4} = -\frac{1}{360} (5p^2 - 3) \leq 0,$$

which leads to  $p \in (\sqrt{15}/5, 1]$ .

(ii) If  $F_p(x) > 0$  for all  $x \in (0, \pi/2)$ , then we have

$$F_p\left(\frac{\pi^-}{2}\right) = \frac{2}{\pi} - \left( \frac{1}{3p^2} \cos \frac{p\pi}{2} + 1 - \frac{1}{3p^2} \right) > 0.$$

From Lemma 1 we see that the function  $p \mapsto F_p(\pi/2^-)$  is decreasing on  $[0, 1]$ , which together with the facts

$$F_{1/2}\left(\frac{\pi^-}{2}\right) = \frac{2}{\pi} - \frac{2}{3}\sqrt{2} + \frac{1}{3} > 0 \text{ and } F_1\left(\frac{\pi^-}{2}\right) = \frac{2}{\pi} - \frac{2}{3} < 0$$

gives that there is a unique number  $p_0 \in (1/2, 1)$  such that  $F_p(\pi/2^-) > 0$  for  $p \in (0, p_0)$  and  $F_p(\pi/2^-) < 0$  for  $p \in (p_0, 1)$ . Solving the equation  $F_p(\pi/2^-) = 0$  for  $p$  by mathematical computer software we find that  $p_0 \approx 0.77086$ .

This completes the proof.  $\square$

**Lemma 3.** *Let  $c \in (0, 3/5]$  and let the sequence  $(a_n(c))$  be defined by*

$$(2.3) \quad a_n(c) = 3 - (2n + 1)c^{n-1}.$$

*Then (i)  $a_n(c) \geq 0$  for  $n \in \mathbb{N}$ ; (ii) for  $n \geq 3$ , we have*

$$1 < \frac{a_{n+1}(c)}{a_n(c)} \leq \frac{a_{n+1}(3/5)}{a_n(3/5)} \leq \frac{11}{5}.$$

*Proof.* (i) We first show that  $a_n(c) \geq 0$  for  $n \in \mathbb{N}$ . A simple computation leads to

$$\begin{aligned} a_{n+1}(c) - a_n(c) &= (2n + 1)c^{n-1} - (2n + 3)c^n = c^{n-1}((2n + 1) - (2n + 3)c) \\ &\geq c^{n-1} \left( (2n + 1) - (2n + 3)\frac{3}{5} \right) = \frac{4}{5}c^{n-1}(n - 1) \geq 0, \end{aligned}$$

which implies that  $a_{n+1}(c) \geq a_n(c) \geq a_1(c) = 0$ .

(ii) Since  $a_1(c) = 0$ ,  $a_2(c) = 3 - 5c \geq 0$ ,  $a_n(c) > 0$  for  $n \geq 3$ , if we can show that the function

$$(c, n) \mapsto \frac{a_{n+1}(c)}{a_n(c)} = \frac{3 - (2n + 3)c^n}{3 - (2n + 1)c^{n-1}}$$

is increasing in  $c$  on  $(0, 3/5]$  and decreasing in  $n \geq 3$ , then we have

$$1 = \frac{a_{n+1}(0)}{a_n(0)} < \frac{a_{n+1}(c)}{a_n(c)} < \frac{a_{n+1}(3/5)}{a_n(3/5)} \leq \frac{a_4(3/5)}{a_3(3/5)} = \frac{11}{5},$$

which is the desired results. Now we prove that  $(c, n) \mapsto a_{n+1}(c)/a_n(c)$  is increasing in  $c$  on  $(0, 3/5]$  for  $n \geq 3$ . Differentiation yields

$$c^{2-n} a_n^2(c) \left( \frac{a_{n+1}(c)}{a_n(c)} \right)' = (4n^2 + 8n + 3) c^n - 3n(2n+3)c + (6n^2 - 3n - 3) := h_n(c),$$

$$h'_n(c) = -n(2n+3)(3 - (2n+1)c^{n-1}) = -n(2n+3)a_n(c) < 0,$$

where the last inequality holds due to  $a_n(c) > 0$  for  $n \geq 3$ . Therefore, we have

$$h_n(c) \geq h_n(3/5) = \left(\frac{3}{5}\right)^n (4n^2 + 8n + 3) + \frac{3}{5} (4n^2 - 14n - 5),$$

which is clearly positive due to that  $h_3(3/5) = 876/125 > 0$  and  $(4n^2 - 14n - 5) = n(4n - 14) - 5 \geq 3$  for  $n \geq 4$ . This reveals that  $h'_n(c) > 0$ , that is,  $(c, n) \mapsto a_{n+1}(c)/a_n(c)$  is increasing in  $c$  on  $(0, 3/5]$ .

On the other hand, we have

$$\begin{aligned} \frac{a_{n+1}(c)}{a_n(c)} - \frac{a_{n+2}(c)}{a_{n+1}(c)} &= c^{n-1} \frac{4c^{n+1} + 6(c-1)^2 n + 15c^2 - 18c + 3}{a_n(c) a_{n+1}(c)} \\ &\geq \frac{c^{n-1}}{a_n(c) a_{n+1}(c)} \left( 4c^{n+1} + 6(c-1)^2 \times 3 + 15c^2 - 18c + 3 \right) \\ &= \frac{c^{n-1}}{a_n(c) a_{n+1}(c)} \left( 4c^{n+1} + 33\left(\frac{7}{11} - c\right)(1-c) \right) > 0, \end{aligned}$$

where the first inequality holds due to  $a_n(c) > 0$  for  $n \geq 3$ , while the last one holds since  $c \in (0, 3/5]$ . This means that  $(c, n) \mapsto a_{n+1}(c)/a_n(c)$  is decreasing with  $n \geq 3$ .

Thus we complete the proof of this assertion.  $\square$

Now we are in a position to prove Theorem 1.

*Proof of Theorem 1.* Expanding in power series gives

$$\begin{aligned} F_p(x) &= \frac{\sin x}{x} - \left( \frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2} \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} - \left( \frac{1}{3p^2} \sum_{n=0}^{\infty} (-1)^n \frac{(px)^{2n}}{(2n)!} + 1 - \frac{1}{3p^2} \right) \\ &= \sum_{n=2}^{\infty} (-1)^n \frac{3 - (2n+1)p^{2n-2}}{3(2n+1)!} x^{2n} \\ &= \frac{3 - 5p^2}{360} x^4 + \sum_{n=3}^{\infty} (-1)^n \frac{a_n(p^2)}{3(2n+1)!} x^{2n}, \end{aligned}$$

where  $a_n(c)$  is defined by (2.3). Considering the function  $f_p(x) = x^{-4}F_p(x)$ , we have

$$(2.4) \quad f_p(x) = x^{-4}F_p(x) = \frac{3 - 5p^2}{360} + \sum_{n=3}^{\infty} (-1)^n \frac{a_n(p^2)}{3(2n+1)!} x^{2n-4},$$

and differentiation yields

$$\begin{aligned} f'_p(x) &= \sum_{n=3}^{\infty} (-1)^n \frac{(2n-4) a_n(p^2)}{3(2n+1)!} x^{2n-5} \\ &: = \sum_{n=3}^{\infty} (-1)^n u_n(x), \end{aligned}$$

where

$$u_n(x) = \frac{(2n-4) a_n(p^2)}{3(2n+1)!} x^{2n-5}.$$

Utilizing Lemma 3 we get that for  $p^2 \in (0, 3/5]$  and  $n \geq 3$ ,

$$\begin{aligned} \frac{u_{n+1}(x)}{u_n(x)} &= \frac{\frac{(2n-2)a_{n+1}(p^2)}{3(2n+3)!} x^{2n-3}}{\frac{(2n-4)a_n(p^2)}{3(2n+1)!} x^{2n-5}} = \frac{1}{(2n-4)(2n+3)} \frac{(2n-2) a_{n+1}(p^2)}{a_n(p^2)} x^2 \\ &< \frac{1}{(2 \times 3 - 4)(2 \times 3 + 3)} \times 1 \times \frac{11}{5} \times \frac{\pi^2}{4} = \frac{11\pi^2}{360} < 1, \end{aligned}$$

which implies that the power series  $\sum_{n=3}^{\infty} (-1)^n u_n(x)$  is a Leibniz type alternating one, and so  $f'_p(x) < 0$  for  $p^2 \in (0, 3/5]$ .

(i) We first prove the second inequality in (1.14) holds, where  $p_1 = \sqrt{15}/5$  is the best. As shown previously, we see that  $f_{p_1}$  is decreasing on  $(0, \pi/2)$ , and therefore,

$$f_{p_1}(x) < f_{p_1}(0^+) = \lim_{x \rightarrow 0^+} (x^{-4} F_p(x)) = \frac{1}{360} (3 - 5p_1^2) = 0,$$

which together with (2.4) yields  $F_{p_1}(x) < 0$  for  $x \in (0, \pi/2)$ .

Next we prove  $p_1 = \sqrt{15}/5$  is the best. If there is another  $p_1^* < p_1$  such that the second inequality in (1.14) holds for  $x \in (0, \pi/2)$ , then by Lemma 2 there must be  $p_1^* \in [p_1, 1]$ , which yields a contradiction. Therefore,  $p_1 = \sqrt{15}/5$  can not be replaced with other smaller ones.

(ii) Now we prove the first inequality in (1.14) holds with the best constant  $p_0 \approx 0.77088$ . Since  $p_0^2 \in (0, 3/5]$ ,  $f_{p_0}$  is also decreasing on  $(0, \pi/2)$ , and so

$$f_{p_0}(x) > f_{p_0}\left(\frac{\pi}{2}^-\right) = \lim_{x \rightarrow \pi/2^-} (x^{-4} F_{p_0}(x)) = \left(\frac{\pi}{2}\right)^{-4} F_{p_0}\left(\frac{\pi}{2}^-\right) = 0,$$

where the last equality is true due to  $p_0$  is the unique root of the equation (1.15) on  $(0, 1)$ . It together with (2.4) gives  $F_{p_0}(x) > 0$  for  $x \in (0, \pi/2)$ .

Lastly, we show that  $p_0$  is the best. Assume that there is another  $p_0^* > p_0$  such that  $F_{p_0^*}(x) > 0$  for  $x \in (0, \pi/2)$ . Then by Lemma 2 there must be  $p_0^* \in [0, p_0]$ , which is clear a contradiction. Consequently,  $p_0$  can not be replaced by other larger numbers.

Thus the proof is complete.  $\square$

**Remark 2.** Application of the conclusion that  $f'_p(x) < 0$  for  $x \in (0, \pi/2)$  if  $p^2 \in (0, 3/5]$  gives  $f_p(\pi/2^-) < f_p(x) < f_p(0^+)$ , that is,

$$\left(\frac{\pi}{2}\right)^{-4} F_p\left(\frac{\pi}{2}^-\right) < x^{-4} F_p(x) < \lim_{x \rightarrow 0^+} x^{-4} F_p(x) = \frac{3 - 5p^2}{360},$$

which can be changed into

$$(2.5) \quad \left(\frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2}\right) + c_0(p) x^4 < \frac{\sin x}{x} < \left(\frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2}\right) + c_1(p) x^4,$$

where  $c_0(p) = (\pi/2)^{-4} F_p(\pi/2^-)$  and  $c_1(p) = (3 - 5p^2)/360$  are the best constants. Then

(i) when  $p = p_1 = \sqrt{15}/5$ , we have

$$c_0(p_1) x^4 + \left(\frac{1}{3p_1^2} \cos p_1 x + 1 - \frac{1}{3p_1^2}\right) < \frac{\sin x}{x} < \left(\frac{1}{3p_1^2} \cos p_1 x + 1 - \frac{1}{3p_1^2}\right),$$

where  $c_0(p_1) = (\pi/2^-)^{-4} F_{p_1}(\pi/2^-) \approx -7.2618 \times 10^{-5}$  and  $c_1(p_1) = 0$  are the best possible constants;

(ii) when  $p = p_0 \approx 0.77086$ , we get

$$\left(\frac{1}{3p_0^2} \cos p_0 x + 1 - \frac{1}{3p_0^2}\right) < \frac{\sin x}{x} < \left(\frac{1}{3p_0^2} \cos p_0 x + 1 - \frac{1}{3p_0^2}\right) + c_1(p_0) x^4,$$

where  $c_0(p_0) = 0$  and  $c_1(p_0) = (3 - 5p_0^2)/360 \approx 8.0206 \times 10^{-5}$  are the best constants.

### 3. PROOF OF THEOREM 2

For proving Theorem 2, we first give the following lemmas.

**Lemma 4.** For  $x \in (0, \infty)$ , the function  $p \mapsto V_p(x)$  defined on  $[0, \infty)$  by

$$V_p(x) = \frac{1}{3p^2} \cosh px + 1 - \frac{1}{3p^2} \text{ if } p \neq 0 \text{ and } V_0(x) = 1 + \frac{x^2}{6}$$

is increasing.

*Proof.* Differentiation yields

$$\begin{aligned} \frac{\partial V_p}{\partial p} &= \frac{1}{3p^3} (px \sinh px - 2 \cosh px + 2) \\ &= \frac{2x}{3p^2} \left( \cosh \frac{px}{2} - \frac{\sinh \frac{px}{2}}{\frac{px}{2}} \right) \sinh \frac{px}{2} > 0, \end{aligned}$$

which completes the proof.  $\square$

**Lemma 5.** Let the function  $G_p$  be defined on  $(0, \infty)$  by

$$G_p(x) = \frac{\sinh x}{x} - \left(\frac{1}{3p^2} \cosh px + 1 - \frac{1}{3p^2}\right) \text{ if } p \neq 0 \text{ and } G_0(x) = \frac{\sinh x}{x} - 1 - \frac{x^2}{6}.$$

(i) If  $G_p(x) < 0$  for all  $x \in (0, \infty)$ , then  $p \geq 1$ .

(ii) If  $G_p(x) > 0$  for all  $x \in (0, \pi/2)$ , then  $p \leq p_1 = \sqrt{15}/5 \approx 0.77460$ .

*Proof.* In order to prove the desired results, we need the following two relations:

$$(3.1) \quad \lim_{x \rightarrow 0} \frac{G_p(x)}{x^4} = -\frac{1}{360} (5p^2 - 3),$$

$$(3.2) \quad \lim_{x \rightarrow \infty} \frac{G_p(x)}{e^{px}} = \begin{cases} -\frac{1}{6p^2} & \text{if } p > 1, \\ -\frac{1}{6} & \text{if } p = 1, \\ \infty & \text{if } 0 < p < 1, \\ \infty & \text{if } p = 0. \end{cases}$$

The first one follows by expanding in power series:

$$G_p(x) = -\frac{1}{360} (5p^2 - 3) x^4 + o(x^6).$$



To obtain the second one, it needs to note that

$$e^{-px}G_p(x) = e^{(1-p)x}\frac{1-e^{-2x}}{2x} - \frac{1}{3p^2}\frac{1-e^{-2px}}{2} - \left(1 - \frac{1}{3p^2}\right)e^{-px},$$

which gives (3.2).

(i) If  $G_p(x) < 0$  for all  $x \in (0, \infty)$ , then we have  $\lim_{x \rightarrow 0} x^{-4}G_p(x) \leq 0$  and  $\lim_{x \rightarrow \infty} e^{-px}G_p(x) \leq 0$ . These together with (3.1) and (3.2) give  $p \geq 1$ .

(ii) If  $G_p(x) > 0$  for all  $x \in (0, \infty)$ , then we have  $\lim_{x \rightarrow 0} x^{-4}G_p(x) \geq 0$  and  $\lim_{x \rightarrow \infty} e^{-px}G_p(x) \geq 0$ . These together with (3.1) and (3.2) indicate  $p \leq p_1$ .  $\square$

We now can prove Theorem 2.

*Proof of Theorem 2.* The necessity follows by Lemma 5. To prove the sufficiency, we expanding  $G_p(x)$  in power series to get

$$\begin{aligned} G_p(x) &= \frac{\sinh x}{x} - \left(\frac{1}{3p^2} \cosh px + 1 - \frac{1}{3p^2}\right) \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} - \left(\frac{1}{3p^2} \sum_{n=0}^{\infty} \frac{(px)^{2n}}{(2n)!} + 1 - \frac{1}{3p^2}\right) \\ &= \sum_{n=2}^{\infty} \frac{3 - (2n+1)p^{2n-2}}{3(2n+1)!} x^{2n} = \sum_{n=2}^{\infty} \frac{a_n(p^2)}{3(2n+1)!} x^{2n}. \end{aligned}$$

It is derived from Lemma 3 that  $a_n(p^2) \geq 0$  if  $0 < p \leq \sqrt{15}/5$ , and clearly,  $a_n(p^2) < 0$  if  $p \geq 1$ .  $\square$

#### 4. APPLICATIONS

As simple applications of main results, we present some precise estimations for certain special functions and constants in this section.

The sine integral is defined by

$$\text{Si}(t) = \int_0^t \frac{\sin x}{x} dx.$$

Some estimates for sine integral can be seen [33], [34], [35], [26], [9]. Now we give a new result.

**Proposition 1.** For  $t \in (0, \pi/2)$  and  $p \in (0, \sqrt{15}/5]$ , we have

$$(4.1) \quad \frac{\sin pt}{3p^3} + \left(1 - \frac{1}{3p^2}\right)t + \frac{c_0(p)}{5}t^5 < \text{Si}(t) < \frac{\sin pt}{3p^3} + \left(1 - \frac{1}{3p^2}\right)t + \frac{c_1(p)}{5}t^5,$$

where  $c_0(p) = (\pi/2)^{-4}F_p(\pi/2^-)$  and  $c_1(p) = (3 - 5p^2)/360$ , here  $F_p(\pi/2^-)$  is defined by (1.15). Particularly, putting  $p = 0^+, 2/3$ , we have

$$(4.2) \quad t - \frac{1}{18}t^3 + \frac{2\pi^3 - 48\pi + 96}{15\pi^5}t^5 < \text{Si}(t) < t - \frac{1}{18}t^3 + \frac{1}{600}t^5,$$

$$(4.3) \quad \frac{9}{8}\sin \frac{2t}{3} + \frac{1}{4}t + \frac{2(16 - 5\pi)}{5\pi^5}t^5 < \text{Si}(t) < \frac{9}{8}\sin \frac{2t}{3} + \frac{1}{4}t + \frac{7}{16200}t^5,$$

and then,

$$1.3705 \approx \frac{2}{5}\pi - \frac{1}{360}\pi^3 + \frac{1}{5} < \text{Si}\left(\frac{\pi}{2}\right) < \frac{1}{2}\pi - \frac{1}{144}\pi^3 + \frac{1}{19200}\pi^5 \approx 1.3714,$$

$$1.3706 \approx \frac{1}{16}\pi + \frac{9\sqrt{3}}{16} + \frac{1}{5} < \text{Si}\left(\frac{\pi}{2}\right) < \frac{1}{8}\pi + \frac{7}{518400}\pi^5 + \frac{9\sqrt{3}}{16} \approx 1.3711.$$

*Proof.* Integrating each sides in (2.5) over  $[0, t]$  yields (4.1). Taking the limits of the left and right hand sides in (4.1) as  $p \rightarrow 0^+$  gives (4.2), and putting  $p = 2/3$  in (4.1) leads to (4.3). Substituting  $t = \pi/2$  into (4.2) and (4.3) we get the last two approximations of  $\text{Si}(\pi/2)$ .  $\square$

It is known that

$$\int_0^\infty \frac{x}{\sinh x} dx = \frac{1}{2} \psi'(\tfrac{1}{2}) = \frac{\pi^2}{4},$$

where  $\psi'$  is the tri-gamma function defined by

$$\psi'(t) = \int_0^\infty \frac{x e^{-tx}}{1 - e^{-x}} dx.$$

We define

$$Sh(t) = \int_0^t \frac{x}{\sinh x} dx.$$

Then by (1.16) we have

$$\frac{3}{\cosh x + 2} < \frac{x}{\sinh x} < \frac{9}{5 \cosh(\sqrt{15}x/5) + 4}.$$

Integrating over  $[0, t]$  and calculating lead to

**Proposition 2.** *For  $t > 0$ , we have*

$$(4.4) \quad \sqrt{3} \ln \frac{e^t - \sqrt{3} + 2}{e^t + \sqrt{3} + 2} - \sqrt{3} \ln(2 - \sqrt{3}) < Sh(t) < 2\sqrt{15} \arctan\left(\frac{5}{3}e^{\sqrt{15}t/5} + \frac{4}{3}\right) - 2\sqrt{15} \arctan 3.$$

*In particular, we have*

$$4.5621 \approx 2\sqrt{3} \ln(2 + \sqrt{3}) < \psi'(\tfrac{1}{2}) < 2\sqrt{15}\pi - 4\sqrt{15} \arctan 3 \approx 4.9845.$$

The Catalan constant [36]

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159655941772190...$$

is a famous mysterious constant appearing in many places in mathematics and physics. Its integral representations contain the following [37]

$$(4.5) \quad G = \int_0^1 \frac{\arctan x}{x} dx = \frac{1}{2} \int_0^{\pi/2} \frac{x}{\sin x} dx = \frac{\pi^2}{16} - \frac{\pi}{4} \ln 2 + \int_0^{\pi/4} \frac{x^2}{\sin^2 x} dx.$$

We present an estimation for  $G$  below.

**Proposition 3.** *We have*

$$(4.6) \quad 0.91586 \approx \frac{\sqrt{15}}{4} \ln \frac{4 \cos \frac{\sqrt{15}\pi}{10} + 3 \sin \frac{\sqrt{15}\pi}{10} + 5}{4 \cos \frac{\sqrt{15}\pi}{10} - 3 \sin \frac{\sqrt{15}\pi}{10} + 5} < G < \frac{\sqrt{15}}{5} \ln \frac{11\sqrt{2-\sqrt{2}}+3\sqrt{15}\sqrt{\sqrt{2}+2}+32}{11\sqrt{2-\sqrt{2}}-3\sqrt{15}\sqrt{\sqrt{2}+2}+32} \approx 0.91675,$$

*Proof.* From the fourth and fifth inequalities in (1.17) we obtain that for  $x \in (0, \pi/2)$ , the two-side inequality

$$\frac{1}{\frac{5}{9} \cos \frac{\sqrt{15}x}{5} + \frac{4}{9}} < \frac{x}{\sin x} < \frac{1}{\frac{16}{27} \cos \frac{3x}{4} + \frac{11}{27}}$$

holds true. Integrating both sides over  $[0, \pi/2]$  yields

$$\int_0^{\pi/2} \frac{dx}{\frac{5}{9} \cos \frac{\sqrt{15}x}{5} + \frac{4}{9}} < \int_0^{\pi/2} \frac{x}{\sin x} dx < \int_0^{\pi/2} \frac{dx}{\frac{16}{27} \cos \frac{3x}{4} + \frac{11}{27}}.$$

Direct computations give

$$\begin{aligned} \int_0^{\pi/2} \frac{dx}{\frac{5}{9} \cos \frac{\sqrt{15}x}{5} + \frac{4}{9}} &= \frac{\sqrt{15}}{2} \ln \frac{4 \cos \frac{\sqrt{15}\pi}{10} + 3 \sin \frac{\sqrt{15}\pi}{10} + 5}{4 \cos \frac{\sqrt{15}\pi}{10} - 3 \sin \frac{\sqrt{15}\pi}{10} + 5} \approx 1.8317, \\ \int_0^{\pi/2} \frac{dx}{\frac{16}{27} \cos \frac{3x}{4} + \frac{11}{27}} &= \frac{2\sqrt{15}}{5} \ln \frac{11\sqrt{2-\sqrt{2}}+3\sqrt{15}\sqrt{\sqrt{2}+2}+32}{11\sqrt{2-\sqrt{2}}-3\sqrt{15}\sqrt{\sqrt{2}+2}+32} \approx 1.8335 \end{aligned}$$

Utilizing the the second formula in (4.5) (4.4) follows.  $\square$

We close this paper by giving some inequalities for bivariate means.

The Schwab-Borchardt mean of two numbers  $a \geq 0$  and  $b > 0$ , denoted by  $SB(a, b)$ , is defined as [38, Theorem 8.4], [39, 3, (2.3)]

$$SB(a, b) = \begin{cases} \frac{\sqrt{b^2-a^2}}{\arccos(a/b)} & \text{if } a < b, \\ a & \text{if } a = b, \\ \frac{\sqrt{a^2-b^2}}{\operatorname{arccosh}(a/b)} & \text{if } a > b. \end{cases}$$

The properties and certain inequalities involving Schwab-Borchardt mean can be found in [40], [41]. We now establish a new inequality for this mean.

For  $a < b$ , letting  $x = \arccos(a/b)$  in the fourth inequality of (1.17) and using half-angle and triple-angle formulas for cosine function, and multiplying two sides by  $b$ , we get

$$\begin{aligned} SB(a, b) &\geq \frac{16}{27}b \sqrt{\frac{1 + \sqrt{\frac{1+4(a/b)^3-3a/b}{2}}}{2}} + \frac{11}{27}b \\ &= \frac{8}{27} \left( \sqrt{2(b-2a)^2(a+b) + 2b^{3/2}} \right)^{1/2} b^{1/4} + \frac{11}{27}b. \end{aligned}$$

For  $a > b$ , letting  $x = \operatorname{arccosh}(a/b)$  in the inequality connecting the fourth and sixth members of (1.17) and using half-angle and triple-angle formulas for hyperbolic cosine function, and multiplying two sides by  $b$ , we get the same inequality as above.

**Proposition 4.** *For  $a, b > 0$ , we have*

$$(4.7) \quad SB(a, b) \geq \frac{8\sqrt{2}}{27} \left( |b-2a| \sqrt{\frac{a+b}{2}} + b^{3/2} \right)^{1/2} b^{1/4} + \frac{11}{27}b.$$

**Remark 3.** *From the inequality (4.7), it is easy to get*

$$SB(a, b) \geq \frac{11+8\sqrt{2}}{27}b \approx 0.82643 \times b$$

due to  $|b-2a| \geq 0$ . It seems to new and interesting.

For  $a, b > 0$ , with  $x = (1/2) \ln(a/b)$ , we have

$$\frac{\sinh x}{x} = \frac{L(a, b)}{G(a, b)}, \quad \cosh px = \frac{(a^p + b^p)/2}{(\sqrt{ab})^p} = \frac{A_p^p(a, b)}{G^p(a, b)},$$

and by Theorem 2 we immediately get the following

**Proposition 5.** *For  $a, b > 0$  with  $a \neq b$ , the double inequality*

$$(4.8) \quad \frac{5}{9} A_{\sqrt{15}/5}^{\sqrt{15}/5} G^{1-\sqrt{15}/5} + \frac{4}{9} G < L < \frac{1}{3} A + \frac{2}{3} G$$

*holds with the best constants  $p_1 = \sqrt{15}/5$  and 1. And, the function*

$$p \mapsto \frac{1}{3p^2} A_p^p G^{1-p} + \left(1 - \frac{1}{3p^2}\right) G \text{ if } p \neq 0 \text{ and } G + \frac{(\ln b - \ln a)^2}{24} G \text{ if } p = 0$$

*is increasing on  $\mathbb{R}$ .*

**Remark 4.** *Accordingly, Corollary 2 can be changed into the chain of inequalities for means:*

$$\begin{aligned} A_{1/\sqrt{3}}^{1/\sqrt{3}} G^{1-1/\sqrt{3}} &< \frac{3}{4} A_{2/3}^{2/3} G^{1/3} + \frac{1}{4} G < \frac{2}{3} A_{1/\sqrt{2}}^{1/\sqrt{2}} G^{1-1/\sqrt{2}} + \frac{1}{3} G \\ &< \frac{16}{27} A_{3/4}^{3/4} G^{1/4} + \frac{11}{27} G < \frac{5}{9} A_{\sqrt{15}/5}^{\sqrt{15}/5} G^{1-\sqrt{15}/5} + \frac{4}{9} G < L \\ &< \frac{1}{3} A + \frac{2}{3} G < \frac{1}{2} A_{1/\sqrt{3}}^{2/\sqrt{3}} G^{1-2/\sqrt{3}} + \frac{1}{2} G. \end{aligned}$$

**Remark 5.** *In [19], Yang obtained a sharp lower bound  $A_{q_0}^{1/(3q_0)} G^{1-1/(3q_0)}$  for the logarithmic mean  $L$ , where  $q_0 = 1/\sqrt{5}$ , and pointed out that this one seems to superior to most of known ones. Now, we derive a new sharp lower bound  $\frac{5}{9} A_{p_1}^{p_1} G^{1-p_1} + \frac{4}{9} G$  for  $L$ , where  $p_1 = \sqrt{15}/5$ . We claim that the latter is better than the former. In fact, we have*

$$(4.9) \quad L > \frac{5}{9} A_{p_1}^{p_1} G^{1-p_1} + \frac{4}{9} G > A_q^{1/(3q)} G^{1-1/(3q)}$$

*if and only if  $q \geq q_0 = 1/\sqrt{5}$ . In order for the second inequality in (4.9) to hold, it suffices that for  $x > 0$*

$$D(x) = \frac{1}{3p_1^2} \cosh p_1 x + 1 - \frac{1}{3p_1^2} - (\cosh qx)^{1/(3q^2)} > 0$$

*if and only if  $q \geq q_0 = 1/\sqrt{5}$ .*

*The necessity can be obtained by  $\lim_{x \rightarrow 0} x^{-4} D(x) \geq 0$ , which follows by expanding in power series*

$$D(x) = \frac{1}{72} x^4 (p_1^2 + 2q^2 - 1) + o(x^6).$$

*Then,  $q \geq \sqrt{(1 - p_1^2)/2} = 1/\sqrt{5}$ .*

*Since  $q \mapsto (\cosh qx)^{1/(3q^2)}$  is decreasing on  $(0, \infty)$  proved in [19, Lemma 2], to prove  $D(x) \geq 0$  if  $q \geq q_0$ , it suffices to show that  $D(x) \geq 0$  when  $q = q_0$ . Differentiation yields*

$$\begin{aligned} D'(x) &= \frac{1}{3p_1} \sinh p_1 x - \frac{1}{3q_0} \cosh^{1/(3q_0^2)-1} q_0 x \sinh q_0 x \\ &= \frac{\sinh q_0 x}{3q_0} \left( \frac{q_0 \sinh p_1 x}{p_1 \sinh q_0 x} - \cosh^{1/(3q_0^2)-1} q_0 x \right) \\ &= \frac{\sinh q_0 x}{3q_0} \times L \left( \frac{q_0 \sinh p_1 x}{p_1 \sinh q_0 x}, \cosh^{1/(3q_0^2)-1} q_0 x \right) \times D_1(x), \end{aligned}$$

*where*

$$D_1(x) = \ln \left( \frac{q_0 \sinh p_1 x}{p_1 \sinh q_0 x} \right) - \left( \frac{1}{3q_0^2} - 1 \right) \ln \cosh q_0 x.$$

Differentiating  $D_1(x)$  gives

$$D_1'(x) = \frac{D_2(x)}{6q_0 \sinh 2q_0 x \sinh p_1 x},$$

where

$$\begin{aligned} D_2(x) &= 4(-\sinh p_1 x \sinh^2 q_0 x - 3q_0^2 \cosh^2 q_0 x \sinh p_1 x \\ &\quad + 3q_0^2 \sinh p_1 x \sinh^2 q_0 x + 3p_1 q_0 \cosh p_1 x \cosh q_0 x \sinh q_0 x). \end{aligned}$$

Utilizing "product into sum" formulas and expanding in power series lead to

$$\begin{aligned} D_2(x) &= -2(6q_0^2 - 1) \sinh p_1 x + (3p_1 q_0 - 1) \sinh(p_1 x + 2q_0 x) \\ &\quad + (3p_1 q_0 + 1) \sinh(2q_0 x - p_1 x) \\ &= \sum_{n=1}^{\infty} d_n \frac{p_1^{2n-1} x^{2n-1}}{(2n-1)!}, \end{aligned}$$

where

$$d_n = (3p_1 q_0 - 1) \left(1 + \frac{2q_0}{p_1}\right)^{2n-1} + (3p_1 q_0 + 1) \left(\frac{2q_0}{p_1} - 1\right)^{2n-1} - 2(6q_0^2 - 1)$$

Now we show that  $d_n \geq 0$  for  $n \geq 1$ . A simple verification yields  $d_1 = d_2 = 0$ ,  $d_3 = 64/45 > 0$ . Suppose that  $d_n > 0$  for  $n > 3$ , that is,

$$(3p_1 q_0 + 1) \left(\frac{2q_0}{p_1} - 1\right)^{2n-1} > 2(6q_0^2 - 1) - (3p_1 q_0 - 1) \left(1 + \frac{2q_0}{p_1}\right)^{2n-1}.$$

Then,

$$\begin{aligned} d_{n+1} &= (3p_1 q_0 - 1) \left(1 + \frac{2q_0}{p_1}\right)^{2n+1} + (3p_1 q_0 + 1) \left(\frac{2q_0}{p_1} - 1\right)^{2n+1} - 2(6q_0^2 - 1) \\ &> (3p_1 q_0 - 1) \left(1 + \frac{2q_0}{p_1}\right)^{2n+1} + \left(2(6q_0^2 - 1) - (3p_1 q_0 - 1) \left(1 + \frac{2q_0}{p_1}\right)^{2n-1}\right) \\ &\quad \times \left(\frac{2q_0}{p_1} - 1\right)^2 - 2(6q_0^2 - 1) \\ &= \frac{8q_0}{p_1^2} \left(p_1 (3p_1 q_0 - 1) \left(1 + \frac{2q_0}{p_1}\right)^{2n-1} - (6q_0^2 - 1)(p_1 - q_0)\right). \end{aligned}$$

Since  $(3p_1 q_0 - 1) = (3\sqrt{3} - 5)/5 > 0$ , using binomial expansion we get

$$\begin{aligned} \frac{p_1^2}{8q_0} d_{n+1} &> p_1 (3p_1 q_0 - 1) \left(1 + (2n-1) \frac{2q_0}{p_1}\right) - (6q_0^2 - 1)(p_1 - q_0) \\ &= 2q_0 (3p_1 q_0 - 1)(2n-1) + q_0 (3p_1^2 + 6q_0^2 - 6p_1 q_0 - 1) \\ &= 2q_0 (3p_1 q_0 - 1)(2n-1) - 2q_0 (3p_1 q_0 - 1) \\ &= 4q_0 (3p_1 q_0 - 1)(n-1) > 0, \end{aligned}$$

where the third equality holds due to  $3p_1^2 + 6q_0^2 = 3$ . By mathematical induction, we have proven  $D_2(x) \geq 0$  for  $n \geq 1$ . It follows that  $D_1'(x) > 0$ , which means that  $D_1$  is increasing on  $(0, \infty)$ , and then  $D_1(x) > \lim_{x \rightarrow 0^+} D(x) = 0$ . This in turn implies that  $D$  is increasing on  $(0, \infty)$ , and therefore,  $D(x) \geq D(0^+) = 0$ , which proves the sufficiency.

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POWER SUPPLY SERVICE CENTER, ZPEPC ELECTRIC POWER RESEARCH INSTITUTE, HANGZHOU, ZHEJIANG, CHINA, 310009

E-mail address: yzhkm@163.com